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Dispersion Laws for Goldstone Bosons in a Color Superconductor

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Abstract

The effective action for Goldstone bosons in the color-flavor locking phase of dense QCD is analyzed. Interaction terms and higher derivatives in the effective action appear to be controlled by different scales. At energies of order of the superconducting gap, the derivative expansion breaks down, while interactions still remain suppressed. The effective action valid at energies and momenta comparable to the gap is derived. Dispersion laws following from this action are such that the energy of Goldstone bosons is always smaller than the gap in the quasiparticle spectrum, and Goldstone bosons always propagate without damping.

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1 Introduction

Cold and dense strongly interacting matter is expected to be in a color superconducting state [1] at sufficiently high baryon density [2, 3]. The color superconducting phase is characterized by diquark condensate, the actual structure of which depends on the number of quark species whose masses are comparable to the gap in the quasiparticle spectrum. If three flavors can be considered light, which is definitely true at very high density [4, 5], the preferred ordering is color-flavor locking (CFL) [6]:

$$\left\langle \psi_a^i \psi_b^j \right\rangle \propto \varepsilon^{ijk} \varepsilon_{abk}, \tag{1.1}$$

where i, j and a, b are the color and the flavor indices, respectively.

The diquark condensate in the CFL phase breaks color $SU(3)_C$ and vector $SU(3)_V$ symmetries of QCD to the diagonal subgroup, and all gluons except one acquire masses via the Higgs mechanism. Since one of the generators of $SU(3)_V$ is the electric charge, the photon mixes with one of the gluons to produce a massless gauge field. The baryon number $U(1)_B$, axial $U(1)_A$ and chiral $SU(3)_A$ symmetries are broken by the diquark condensate to $\mathbf{Z}_2$.

Apart from the gauge boson of unbroken $U(1)$, the only light degrees of freedom in the CFL phase at energies well below the superconducting gap are Goldstone bosons of the broken global symmetries. The spectrum of Goldstone bosons closely resembles the one in the hadronic phase [6, 7, 8], and, following [9], the bosons associated with chiral symmetry below will be referred to as pions and the boson associated with $U(1)_A$ will be referred to as η'. Actually, $U(1)_A$ is not exactly a symmetry of QCD because of the chiral anomaly, and η' is not exactly massless even in the chiral limit, but, if the chemical potential is much larger than Λ_{QCD}, the effects of anomalous violation of $U(1)_A$ are negligibly small [10], and then η' is sufficiently light to be treated as a Goldstone boson.

The quark masses violate chiral symmetry and, consequently, make pions and η' (but not to the baryon number boson) massive. A usual way to deal with the quark masses is to treat them as a perturbation*. As a first approximation, quark masses can be neglected, and I will concentrate on the chiral limit throughout this paper.

In the chiral limit, the effective action for Goldstone bosons depends only on the derivatives of the fields. Terms with at most two derivatives are fixed by symmetries

*In contrast to the ordinary pion masses at zero density, the squares of the Goldstone boson masses in the CFL phase are proportional to the quark mass squared [6, 8].

[11, 12]:

$$\begin{aligned}\mathcal{L} &= \frac{F_\pi^2}{4}\,\mathrm{tr}\left(\partial_0 U^\dagger \partial_0 U - v_\pi^2\,\partial_i U^\dagger \partial_i U\right) + \frac{1}{2}\left(\partial_0\varphi\partial_0\varphi - v_B^2\,\partial_i\varphi\partial_i\varphi\right) \\ &\quad + \frac{1}{2}\left(\partial_0\vartheta\partial_0\vartheta - v_{\eta'}^2\,\partial_i\vartheta\partial_i\vartheta\right),\end{aligned} \tag{1.2}$$

where

$$U = \exp\left(i\lambda^A\pi^A/F_\pi\right),$$

λ^A are generators of $SU(3)$, and the fields φ and ϑ describe the baryon number and the η' bosons.

At large baryon density, the asymptotic freedom allows to compute the parameters of the low-energy effective theory directly from QCD in a systematic way. In particular, all the parameters in (1.2) [9, 13, 14], as well as corrections to the effective Lagrangian due to quark masses [9, 15, 16, 14, 17, 18], have been calculated. The velocities of all Goldstone bosons appear to be equal to the velocity of sound in the relativistic fluid: $v^2 = 1/3$, and the pion form-factor appears to be very large: $F_\pi \sim \mu$. An important implication of this fact is that pion-pion interactions are weak at low energies, because interaction vertices are suppressed by powers of μ, as can be seen expanding (1.2) in the pion fields. There are two scales in the problem, however: the chemical potential μ and the superconducting gap Δ, and $\Delta \ll \mu$. Higher-derivative corrections to (1.2), in general, are suppressed only by powers of the smaller scale Δ, but the interaction terms for all Goldstone bosons appear to be suppressed by μ at any order of the derivative expansion, as will be shown in Sec. 2. So, the Goldstone bosons remain weakly interacting at energies comparable to the gap, while the kinetic terms in the effective Lagrangian at these energies are significantly changed by derivative corrections. This means that the dynamics of the Goldstone bosons remains relatively simple, and the effective theory for the Goldstone modes can be extended to the energies comparable to the gap by taking into account all orders of the derivative expansion. It should be mentioned that the effective theory becomes essentially non-local at energies and momenta of order of the gap.

2 General structure of the effective action

Let me sketch how the effective action for the Goldstone bosons can be derived from the first principles. At asymptotically large chemical potential, the dominant inter-

action between quarks is the one-gluon exchange. The quark action has the form:

$$S_q = \int d^4x\,\bar\psi(i\partial\!\!\!/ + \gamma^0\mu)\psi - \frac{g^2}{2}\int d^4x d^4y\, j^A_\mu(x) D^{\mu\nu}(x-y) j^A_\nu(y), \tag{2.1}$$

$$j^A = \bar\psi_a \gamma_\mu T^A \psi_a. \tag{2.2}$$

Since condensation occurs in the diquark channel, it is natural to represent the non-local four-fermion vertex in (2.1) in the form

$$\psi^\dagger_A \psi^\dagger_B G^{AB,CD} \psi_C \psi_D,$$

where color, flavor, and Dirac indices and space-time coordinates of the quark fields are collectively denoted by one index. The four-fermion interaction can be bosonized by Habbard-Stratonovich transformation, which requires the introduction of the collective field $\Sigma^{ij}_{ab}(x,y)$ with the diquark quantum numbers. The action

$$S'_q = \int d^4x\,\bar\psi(i\partial\!\!\!/ + \gamma^0\mu)\psi + \psi_A \Sigma^{AB}\psi_B + \psi^\dagger_A \Sigma^{\dagger\,AB}\psi^\dagger_B + \frac{2}{g^2}\,\Sigma^{\dagger\,AB} G^{-1}_{AB,CD}\Sigma^{CD}. \tag{2.3}$$

is then equivalent to (2.1) after elimination of Σ via its equations of motion. Alternatively, integration over fermions yields the effective action for Σ. Schematically,

$$S_{\rm eff} = -i\,{\rm Tr}\ln(i\partial\!\!\!/ + \gamma^0\mu + \Sigma) + \frac{2}{g^2}\,\Sigma^\dagger G^{-1}\Sigma. \tag{2.4}$$

The vacuum expectation value of Σ, which minimizes this action, determines the superconducting gap. The dominant gap is scalar, parity-even and anti-symmetric in color [6]:

$$\left\langle \Sigma^{ij}_{ab\,\alpha\beta}\right\rangle = (C\gamma^5)_{\alpha\beta}\boldsymbol{\Delta}^{ij}_{ab}, \tag{2.5}$$

$$\boldsymbol{\Delta}^{ij}_{ab} = \Delta\varepsilon^{ijk}\varepsilon_{abk}, \tag{2.6}$$

where $C = i\gamma^2\gamma^0$ is the charge conjugation matrix. The value of the gap Δ is proportional to $\,{\rm e}^{-{\rm const}/g}\mu$ [19, 20, 21, 22]. The vacuum expectation value of Σ contains, in principle, other Dirac, spin, and color-flavor structures, but all of them were found to be suppressed at weak coupling [23, 10, 24, 25, 26].

The effective potential for Σ may have rather complicated structure, which is discussed in [27, 10]. But, irrespectively of the details of this structure, the minima of the potential are necessarily degenerate due to the global symmetries of the original quark Lagrangian. Any transformation of the form

$$\Sigma \to \,{\rm e}^{2i\phi+2i\gamma^5\theta} V^T \Sigma V, \tag{2.7}$$

where

$$V = \mathrm{e}^{i\gamma^5 \lambda^A \pi^A / F_\pi},$$

leaves the action (2.4) invariant. The second term in the action is invariant because of the $SU(3)_A \times U(1)_B \times U(1)_A$ symmetry of the gluon vertex. In the first term, the transformation of the gap can be compensated by the rotation of the quark fields:

$$\psi \to \mathrm{e}^{-i\phi - i\gamma^5\theta} V^\dagger \psi. \tag{2.8}$$

Since the Goldstone modes correspond to motion along the degenerate minima of the effective potential, the effective action for the Goldstone bosons is obtained by freezing the modulus of Σ at its vacuum expectation value and allowing the phases in (2.7) to depend on time and on space coordinates. Since the gluon vertex is invariant even under local $SU(3)_A \times U(1)_B \times U(1)_A$ transformations, the effective action comes entirely from the fermion determinant:

$$S = -i \,\mathrm{Tr} \ln \left[i\partial\!\!\!/ + \gamma^0 \mu + \mathrm{e}^{i\phi(x) + i\gamma^5\theta(x)} V^T(x) \left\langle \Sigma(x,y) \right\rangle V(y) \, \mathrm{e}^{i\phi(y) + i\gamma^5\theta(y)} \right]. \tag{2.9}$$

The fields ϕ and θ here differ from φ and ϑ in (1.2) by normalization factors. Alternatively, Σ can always be aligned in a fixed direction by rotation of the quark fields. Then the Goldstone fields appear in the derivative term:

$$S = \mathrm{Tr} \ln \left(iD\!\!\!\!/ + \gamma^0 \mu + \left\langle \Sigma(x,y) \right\rangle \right), \tag{2.10}$$

$$D_\nu = \partial_\nu + V^\dagger \partial_\mu V + i\partial_\nu \phi + i\gamma^5 \partial_\nu \theta. \tag{2.11}$$

This form of the effective action was the starting point of Ref. [9].

An important feature of the effective action for the Goldstone bosons is its independence of a particular form of the gluon vertex. If the one-gluon exchange were replaced by any $SU(3)_A \times U(1)_A \times U(1)_B$ invariant four-quark interaction, the effective action would not change. In particular, the NJL model of [6] will lead to the same effective theory for the Goldstone bosons as the one-gluon exchange under the assumption of validity of the mean field approximation, in other words, if the fluctuations of $|\Sigma|$ are neglected. This is also true for the instanton-based models [2, 3, 28, 29] with the exception that the quark vertex induced by instantons violates $U(1)_A$ and, hence, η' should not be considered as a low-energy excitation. Thus, the dynamics of the Goldstone bosons in the CFL phase is, to a large extent, model independent, and the effective action (2.10) gives a reasonable description of the low-energy excitations in the CFL phase independently of the underlying quark interactions, as long as fluctuations of $|\Sigma|$ are not too strong.

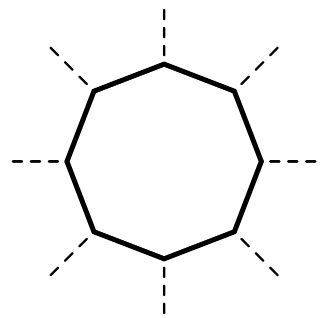

Figure 1: Typical diagram that contributes to the effective action.

I would like to stress the analogy between the effective theory for the Goldstone bosons in the color superconductor (2.9) and the model of [30, 31], in which the chiral Lagrangian for ordinary pions is induced by quark determinant. The pion fields enter through chiral rotations of the constituent quark mass. The constituent mass is the only dimensionful parameter in the model [30, 31] and sets the scale on which the derivative expansion of the fermion determinant breaks down. In the case of the Goldstone bosons in a color superconductor, the situation is different, as there are two well separated scales: the gap Δ and the chemical potential μ, and $\mu \gg \Delta$. The presence of a small parameter Δ/μ leads to important simplifications. The derivative expansion is still controlled by the mass gap, but, as will be shown shortly, the interaction terms in the effective action are all suppressed by $1/\mu$.

The derivative expansion of (2.10) is given by a sum of quark loop diagrams of the type shown in Fig. 1. The diagram with n external legs produces operators of dimension n (and higher, if external momenta are taken into account). Coefficients before dimension d operators are dimension $(4-d)$ combinations of μ and Δ. But the dependence on μ can be easily found and actually is the same at any order of the derivative expansion. The reason is that, at large chemical potential, the momentum integral in the diagram 1 is dominated by poles the quark propagators would have on the Fermi surface in the absence of the gap. The leading order contribution to the momentum integral comes from a thin shell around the Fermi surface:

$$\int \frac{d^4p}{(2\pi)^4} = \frac{\mu^2}{4\pi^3} \int dp_0 dq, \tag{2.12}$$

where $q = |\mathbf{p}| - \mu$. If the integration over p_0 and q converges, the typical contribution comes from $p_0, q \sim \Delta$ and does not depend on μ. The factor of $4\pi\mu^2$ is merely the area of the Fermi surface.

The above arguments show that, up to $O(\Delta/\mu)$ corrections, μ^2 enters the effective action only as an overall factor:

$$S = \mu^2\Delta^2 \int d^4x\, \mathcal{L}\left(\frac{U^\dagger \partial U}{\Delta}, \frac{\partial\phi}{\Delta}, \frac{\partial\theta}{\Delta}; \frac{\partial^2}{\Delta^2}\right). \tag{2.13}$$

The conventional normalization of kinetic terms in the action requires that the fields are rescaled as: $U = \exp(i\lambda^A\pi^A/\mu) = 1 + i\lambda^A\pi^A/\mu + \ldots$, $\phi = \varphi/\mu$ and $\theta = \vartheta/\mu$, after which all interaction terms appear to be suppressed by powers of $1/\mu$. If all energies, momenta, and field strengths are much smaller than μ, the effective action takes the form:

$$S = \int \frac{d^4k}{(2\pi)^4}\left[\pi^A(-k)\Pi\left(\frac{k^2}{\Delta^2}\right)\pi^A(k) + \varphi(-k)K\left(\frac{k^2}{\Delta^2}\right)\varphi(k) + \vartheta(-k)\tilde{K}\left(\frac{k^2}{\Delta^2}\right)\vartheta(k)\right]. \tag{2.14}$$

Further expansion in derivatives yields the low-energy effective Lagrangian (1.2), but the effective action (2.14) is valid also at energies and momenta of order Δ. In principle, interaction terms can be computed perturbatively in Δ/μ.

Because of the lack of Lorentz invariance at finite baryon density, k_0^2 and $\mathbf{k}^2$ enter the propagators of the Goldstone bosons independently. The poles of the propagators determine dispersion relations for the Goldstone bosons:

$$\Pi(\omega(\mathbf{k}),\mathbf{k}) = 0, \qquad K(\omega_s(\mathbf{k}),\mathbf{k}) = 0, \qquad \tilde{K}(\tilde{\omega}_s(\mathbf{k}),\mathbf{k}) = 0 \tag{2.15}$$

The inverse propagators Π, K, and $\tilde{K}$ are computed below in Sec. 4.

3 Interaction of quasiparticles with Goldstone bosons

The gap in (2.3) mixes quarks with anti-quarks. The usual way to deal with that is to treat charge-conjugate quark operators,

$$\psi_C = C\bar{\psi}^T, \tag{3.1}$$

as independent fields. The fermion part of the action (2.3) can be represented in the form

$$S_f = \frac{1}{2}\int d^4x\left[\bar{\psi}(i\partial\!\!\!/ + \gamma^0\mu)\psi + \bar{\psi}_C(i\partial\!\!\!/ - \gamma^0\mu)\psi_C + \bar{\psi}_C\gamma^5\mathbf{\Delta}\psi - \bar{\psi}\gamma^5\mathbf{\Delta}\psi_C\right], \tag{3.2}$$

where Σ is fixed at its vacuum expectation value (2.5).

Since the gap (2.5) does not mix left and right quasiparticles, it is convenient to make one step more and to separate left and right sectors explicitly. Taking into account that the charge conjugate of the right-handed spinor is left-handed:

$$(\psi_C)_L = \frac{1-\gamma^5}{2}\, C\bar{\psi}^T = (\psi_R)_C, \tag{3.3}$$

it is natural to describe left-handed and right-handed quasiparticles by the following four-component spinors

$$\begin{aligned} \chi_L &= \frac{1-\gamma^5}{2}\,\psi + \frac{1+\gamma^5}{2}\,\psi_C\,, \\ \chi_R &= \frac{1+\gamma^5}{2}\,\psi + \frac{1-\gamma^5}{2}\,\psi_C\,. \end{aligned} \tag{3.4}$$

The Lagrangian decouples into two independent pieces in terms of χ_L and χ_R:

$$\mathcal{L}_f = \frac{1}{2}\,\bar{\chi}_L(i\partial\!\!\!/ - \gamma^0\gamma^5\mu - \mathbf{\Delta})\chi_L + \frac{1}{2}\,\bar{\chi}_R(i\partial\!\!\!/ + \gamma^0\gamma^5\mu + \mathbf{\Delta})\chi_R. \tag{3.5}$$

The fields χ_L and χ_R are then treated as independent variables in the path integral.

The interaction of quasiparticles with gluons and Goldstone bosons is described by the Lagrangian

$$\mathcal{L}_f = \frac{1}{2}\,\bar{\chi}_L(iD\!\!\!\!/^L - \gamma^0\gamma^5\mu - \mathbf{\Delta})\chi_L + \frac{1}{2}\,\bar{\chi}_R(iD\!\!\!\!/^R + \gamma^0\gamma^5\mu + \mathbf{\Delta})\chi_R. \tag{3.6}$$

The form of the covariant derivatives D^L and D^R is dictated by transformation laws of χ_L and χ_R under chiral, gauge, and baryon number transformations. As follows from the definition of χ_L, χ_R in terms of the original quark operators, these fields transform as

$$\begin{aligned} \chi_R &\to \mathrm{e}^{-i\gamma^5(\theta+\phi)}\left(\frac{1+\gamma^5}{2}\,U^\dagger\Omega^\dagger + \frac{1-\gamma^5}{2}\,U^T\Omega^T\right)\chi_R, \\ \chi_L &\to \mathrm{e}^{-i\gamma^5(\theta-\phi)}\left(\frac{1+\gamma^5}{2}\,U^*\Omega^T + \frac{1-\gamma^5}{2}\,U\Omega^\dagger\right)\chi_L, \end{aligned} \tag{3.7}$$

where Ω acts on color indices and U describes the rotation in the flavor space. The above transformation laws yield for covariant derivatives:

$$\begin{aligned} D^R_\mu &= \partial_\mu + \frac{1}{2}(L_\mu - L^T_\mu) + \frac{g}{2}(A_\mu - A^T_\mu) \\ &\quad +\gamma^5\left[\frac{1}{2}(L_\mu + L^T_\mu) + \frac{g}{2}(A_\mu + A^T_\mu) + i\partial_\mu\theta + i\partial_\mu\phi\right], \end{aligned} \tag{3.8}$$

$$\begin{aligned} D^L_\mu &= \partial_\mu - \frac{1}{2}(R_\mu - R^T_\mu) + \frac{g}{2}(A_\mu - A^T_\mu) \\ &\quad +\gamma^5\left[\frac{1}{2}(R_\mu + R^T_\mu) - \frac{g}{2}(A_\mu + A^T_\mu) + i\partial_\mu\theta - i\partial_\mu\phi\right], \end{aligned} \tag{3.9}$$

where g is the QCD coupling, A_μ is the anti-Hermitian gluon field, L_μ and R_μ are the left and the right pion currents:

$$L_\mu = U^\dagger \partial_\mu U = \frac{i}{F_\pi}\,\partial_\mu \pi + O(1/\mu^2), \qquad R_\mu = \partial_\mu U U^\dagger = \frac{i}{F_\pi}\,\partial_\mu \pi + O(1/\mu^2). \tag{3.10}$$

The quasiparticle propagators can be found by diagonalization of the Dirac operators in (3.5). The color-flavor structure of the gap matrix can be diagonalized by separating components of χ belonging to the octet and the singlet representations of the unbroken $SU(3)$ group. The projectors on definite representations have the form:

$$\mathbf{P_1}{}^{ij}_{ab} = \frac{1}{3}\,\delta^i_a \delta^j_b, \tag{3.11}$$

$$\mathbf{P_8}{}^{ij}_{ab} = \delta^{ij}\delta_{ab} - \frac{1}{3}\,\delta^i_a \delta^j_b. \tag{3.12}$$

The gap matrix can be decomposed as

$$\boldsymbol{\Delta} = \Delta \mathbf{Q} + 2\Delta \mathbf{P_1}, \tag{3.13}$$

where

$$\mathbf{Q}^{ij}_{ab} = \frac{1}{3}\,\delta^i_a \delta^j_b - \delta^i_b \delta^j_a \tag{3.14}$$

is the square root of $\mathbf{P_8}$:

$$\mathbf{Q}^{ij}_{ab}\mathbf{Q}^{jk}_{bc} = \mathbf{P_8}{}^{ik}_{ac}. \tag{3.15}$$

The Dirac structure of the quasiparticle Lagrangian is diagonalized with the help of the helicity projector:

$$H_\pm = \frac{1 \pm \gamma^0\gamma^5\gamma^i n_i}{2}, \qquad \mathbf{n} = \frac{\mathbf{p}}{|\mathbf{p}|}. \tag{3.16}$$

The propagator of right quasiparticles is

$$S_R(p) = H_+\left(\frac{\not{p}_- - \Delta\mathbf{Q}}{p_-^2 - \Delta^2}\,\mathbf{P_8} + \frac{\not{p}_- - 2\Delta}{p_-^2 - 4\Delta^2}\,\mathbf{P_1}\right) + H_-\left(\frac{\not{p}_+ - \Delta\mathbf{Q}}{p_+^2 - \Delta^2}\,\mathbf{P_8} + \frac{\not{p}_+ - 2\Delta}{p_+^2 - 4\Delta^2}\,\mathbf{P_1}\right), \tag{3.17}$$

where

$$p_\pm = (p_0, \mathbf{p} \pm \mu\mathbf{n}) = (p_0, (|\mathbf{p}| \pm \mu)\mathbf{n}). \tag{3.18}$$

The propagator for the left quasiparticles is obtained by flipping signs of μ and Δ.

Only the first term in the propagator picks up large contributions at the Fermi surface, and the second term will be omitted in most of the calculations below. In principle, a separation of the relevant modes could have been done directly on the level of the action for quasiparticles [32].

4 Effective action and dispersion of the Goldstone bosons

As discussed in Sec. (2), interactions of the Goldstone bosons are suppressed at large chemical potential, so the quadratic terms constitute the most important part of the effective action

$$S = -\frac{i}{2}\,\mathrm{Tr}\ln(i\not{D}^L - \gamma^0\gamma^5\mu - \mathbf{\Delta}) - \frac{i}{2}\,\mathrm{Tr}\ln(i\not{D}^R + \gamma^0\gamma^5\mu + \mathbf{\Delta}). \tag{4.1}$$

The calculation of the quadratic terms in the effective action amounts to evaluation of polarization operators which arise from the derivative expansion of fermion determinants:

$$-\frac{i}{2}\,\mathrm{Tr}\ln(i\not{D} \pm \gamma^0\gamma^5\mu \pm \mathbf{\Delta}) = -\,\mathrm{tr}\,v_\mu\Pi^{\mu\nu}v_\nu - \mathrm{tr}\,a_\mu\Pi^{\mu\nu}a_\nu + s_\mu K^{\mu\nu}s_\nu + \ldots. \tag{4.2}$$

Here, the covariant derivative

$$D_\mu = \partial_\mu + v_\mu + \gamma^5(a_\mu + is_\mu) \tag{4.3}$$

contains $SU(3)$ singlet s_μ and adjoint fields a_μ, v_μ, that obey

$$a_\mu^T = a_\mu, \qquad v_\mu^T = -v_\mu. \tag{4.4}$$

The polarization operators are even functions of μ and Δ and, thus, are the same for the left and the right quasiparticles. It is less obvious that the axial and the vector polarization operators are the same, but it also follows from the symmetry properties (4.4) and the color-flavor structure of the quasiparticle propagator.

Substitution of the explicit expressions for the covariant derivatives (3.8), (3.9) gives for the terms quadratic in currents:

$$S = -\,\mathrm{tr}\,L_\mu\Pi^{\mu\nu}L_\nu - \mathrm{tr}\,R_\mu\Pi^{\mu\nu}R_\nu + \partial_\mu\theta\,K^{\mu\nu}\,\partial_\nu\theta + \partial_\mu\phi\,K^{\mu\nu}\,\partial\phi_\nu + \ldots. \tag{4.5}$$

Consequently, the inverse propagators of the Goldstone bosons are given by longitudinal components of the polarization operators:

$$\Pi(k) = k_\mu\Pi^{\mu\nu}(k)k_\nu, \qquad \tilde{K}(k) = K(k) = k_\mu K^{\mu\nu}(k)k_\nu. \tag{4.6}$$

The same polarization operators describe mass terms for gluons and the mixing between gluons and the Goldstone boson currents. The appearance of mixed terms does not imply mixing of gluons with pions, which is forbidden by parity. A_μ mixes

with the difference of currents $L_\mu - R_\mu$ that is at most quadratic in pion fields, and the mixing describes parity-even interaction:

$$\mathcal{L}_{g\pi\pi} \propto g\mu^2 \,\mathrm{tr}\, A^\nu (L_\mu - R_\mu). \tag{4.7}$$

This term arises because pions are colored as a result of the color-flavor locking. It can be rewritten as

$$\mathcal{L}_{g\pi\pi} \propto g\mu^2 \,\mathrm{tr}\, \partial_\nu U [A^\nu, U^\dagger], \tag{4.8}$$

which is a part of the covariant derivative squared,

$$\mathrm{tr}(D_\mu U)^\dagger D^\mu U, \qquad D_\mu = \partial_\mu + g[A_\mu, \cdot].$$

The appearance of this term means that ordinary derivatives in the effective Lagrangian (1.2) must be replaced by covariant ones, if the gluon fields are taken into account.

The polarization operators are calculated explicitly in the Appendix. Extracting the longitudinal part from Eqs. (A.12)–(A.14) we get, according to (4.6), the inverse pion propagator:

$$\begin{aligned}\Pi(\omega, k) \;&=\; -\frac{\mu^2\Delta^2}{24\pi^2 k}\int_0^1 dx \int_0^k d\alpha\,(\alpha^2-\omega^2)\left[\frac{5}{x(1-x)(\alpha^2-\omega^2)+\Delta^2}\right.\\ &\quad \left. +\frac{4-6x}{x(1-x)(\alpha^2-\omega^2)+(4-3x)\Delta^2}\right],\end{aligned} \tag{4.9}$$

where $\omega = k_0$ and $k = |\mathbf{k}|$. Expansion at low momenta reproduces the result of [9] for the pion form-factor:

$$\begin{aligned}\Pi(\omega, k) \;&=\; \frac{\mu^2}{24\pi^2}\left[\frac{1}{3}\,(21-8\ln 2)\left(\omega^2-\frac{1}{3}\,k^2\right)\right.\\ &\quad \left. +\frac{1}{54\Delta^2}(123-112\ln 2)\left(\omega^4-\frac{2}{3}\,k^2\omega^2+\frac{1}{5}\,k^4\right)+O\left(\frac{1}{\Delta^4}\right)\right].\end{aligned} \tag{4.10}$$

The pole of the propagator determines the dispersion law for pions. At low momenta, pions have the same dispersion relation as relativistic density waves [9]. The corrections tend to decrease the frequency:

$$\omega(k) = \frac{1}{\sqrt{3}}\,k\left[1-\frac{1}{135}\,\frac{123-112\ln 2}{21-8\ln 2}\,\frac{k^2}{\Delta^2}+O\left(\frac{k^4}{\Delta^4}\right)\right] \qquad (k\to 0). \tag{4.11}$$

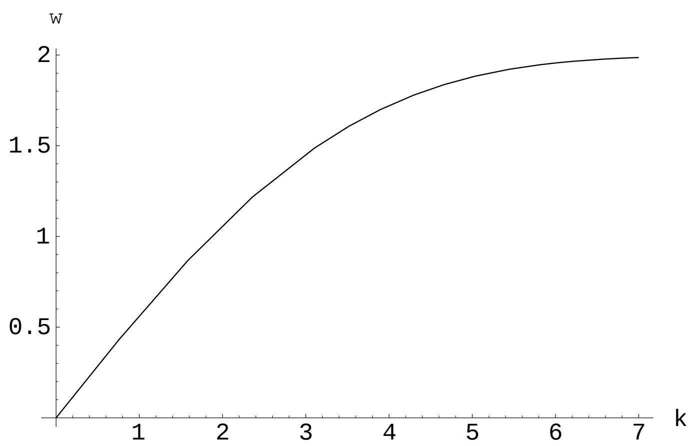

Figure 2: ω as a function of k in the units of Δ.

At higher momenta, the dispersion curve becomes more steep and at $k \to \infty$ the frequency approaches 2Δ from below, since:

$$\Pi(\omega, k) \approx \frac{\mu^2 \Delta^2}{24\pi^2} \left[\frac{10\pi\Delta}{k} \ln\left(\frac{\Delta}{2\Delta - \omega}\right) - 24 \ln\left(\frac{k}{\Delta}\right)\right] \tag{4.12}$$

at $\omega \to 2\Delta$ and $k \to \infty$. Consequently,

$$\omega(k) = 2\Delta - \text{const}\ \Delta \exp\left[-\frac{12k}{5\pi\Delta} \ln\left(\frac{k}{\Delta}\right)\right] \qquad (k \to \infty). \tag{4.13}$$

Here, the momentum k is assumed to be large compared to Δ, but should be much smaller than μ. Actually, this result can be trusted up to $k \sim \sqrt{\Delta\mu}$, after which the approximation (A.7) used in its derivation is no longer valid†.

The dispersion curve for pions is shown in Fig. 2. The energy of a pion never becomes larger than the gap in the quasiparticle spectrum, and the decay of the pion to the constituent quasiparticle-anti-quasiparticle pair is always kinematically forbidden. The pion frequency has no imaginary part, as a result, and pions propagate without damping, unless weak interactions are taken into account.

The energy of a pion can never reach 2Δ, because the polarization operator $\Pi^{\mu\nu}(\omega, \mathbf{k})$ has a logarithmic singularity when the energy approaches the threshold of the pair creation. The singularity comes from the integration over small $\alpha = \mathbf{k}\cdot\mathbf{p}/|\mathbf{p}|$, where $\mathbf{p}$ is the momentum of quasiparticles in the loop. Thus, the threshold singularity is due to creation of an almost on-shell Cooper pair with the momentum of

†I am grateful to D.K. Hong and K. Rajagopal for the discussion of this point.

quasiparticles perpendicular to the momentum of the pion. This singularity compensates for the second term in (4.12), which comes from integration over $\alpha \sim k$, and thus corresponds to creation of the Cooper pair composed of quasiparticles moving in the same direction as the pion. The appearance of the logarithmic singularity is a manifestation of the (2+1)-dimensional nature of the momentum integration near the Fermi surface. In (3+1) dimensions, the square root singularity would arise, which does not blow up at the threshold.

The fact that pions can mix with virtual Cooper pairs might seem puzzling, since a Cooper pair has baryon charge 2. There is no contradiction with baryon charge conservation, however, as we are dealing with the unusual situation in which the baryon symmetry is broken. The vacuum, as well as excited states, are not eigenstates of the baryon charge operator, so the pions do not have definite baryon charge and symmetry arguments do not forbid a non-zero matrix element $\langle \mathrm{vac}|\psi\psi|\pi\rangle$.

The inverse propagator of the singlet Goldstone bosons follows from Eqs. (A.16)–(A.18):

$$\begin{aligned} K(\omega, k) &= -\frac{\mu^2\Delta^2}{\pi^2 k}\int_0^1 dx \int_0^k d\alpha\,(\alpha^2-\omega^2)\left[\frac{2}{x(1-x)(\alpha^2-\omega^2)+\Delta^2}\right. \\ &\quad \left. +\frac{1}{x(1-x)(\alpha^2-\omega^2)+4\Delta^2}\right]. \end{aligned} \tag{4.14}$$

The dispersion relation is qualitatively the same as that for the octet Goldstone bosons. At low momenta:

$$\omega_s(k) = \frac{1}{\sqrt{3}}\,k\left[1-\frac{11}{540}\frac{k^2}{\Delta^2}+O\left(\frac{k^4}{\Delta^4}\right)\right] \qquad (k\to 0). \tag{4.15}$$

At high momenta:

$$\omega_s(k) = 2\Delta - \mathrm{const}\;\Delta\exp\left[-\frac{3k}{\pi\Delta}\,\ln\left(\frac{k}{\Delta}\right)\right] \qquad (k\to\infty). \tag{4.16}$$

5 Discussion

The order parameter of the color superconductivity is the diquark condensate or the conjugate variable, the gap. The orientation of the condensate in the color and flavor space describe the Goldstone modes, which are the only low-energy degrees of freedom away from the phase transition. Power-counting arguments show that the Goldstone bosons are weakly interacting due to the presence of two widely separated

scales in the problem: $\Delta \ll \mu$. The effective action for Goldstone bosons has a form (2.13) and, after appropriate rescaling of the fields, all nonlinearities in it can be omitted. This approximation, of course, implies that the Goldstone fields themselves are not too strong: $|\pi^A| \sim \Delta$. The situation is, to some extent, inverse to what happens at zero chemical potential. Interaction terms and derivative corrections in the ordinary chiral Lagrangian become important more or less at the same scale. The parameter that governs interactions (F_π) may even be smaller (cf. [30, 31]), so the chiral dynamics is essentially non-linear. In the CFL phase, the non-linear effects are much less important than derivative corrections.

The simplicity of the dynamics of the Goldstone bosons allows to go beyond the derivative expansion and to derive the effective action for the Goldstone bosons which contains all powers of derivatives and is valid at energies and momenta of order of the gap. This effective action determines the dispersion laws for the Goldstone bosons. The dispersion of the Goldstone bosons turns out to be similar to the dispersion of phonons in a crystal. In particular, the energy of Goldstone bosons does not grow indefinitely as a function of momentum and is always smaller than the gap in the quasiparticle spectrum. The dispersion curve flattens at large momenta and the group velocity of Goldstone bosons becomes exponentially small.

The deviation of the velocity of pions from the velocity of light by a factor of $1/\sqrt{3}$ has interesting implications for weak interactions in a color superconductor. Pions in a CFL phase are much lighter than in the vacuum: $m_\pi^2 \propto m_s m_{u,d} \Delta^2/\mu^2$ [possibly, up to a factor of $\ln(\mu/\Delta)$] [15, 16, 14, 17, 18]. With $\Delta/\mu \approx 1/10$ taken as an order of magnitude estimate, $m_e \ll m_\pi \ll m_\mu$, so the dominant decay mode of charged pions is

$$\pi^\pm \to e^\pm \nu, \tag{5.1}$$

For this decay to be kinematically allowed, the energy of a pion must at least be larger than the energy of an electron with the same momentum: $\omega(k) > \sqrt{k^2 + m_e^2}$. Since the energy of a pion is bounded by 2Δ, the decay is forbidden at large enough k. In fact, the critical momentum is much smaller than Δ: With the mass of the pion taken into account, the dispersion relation (4.11) becomes

$$\omega(k) = \sqrt{\frac{1}{3}\,k^2 + m_\pi^2}. \tag{5.2}$$

The energy conservation then requires

$$\frac{1}{3}\,k^2 + m_\pi^2 > k^2 + m_e^2,$$

or, neglecting electron mass,

$$k < k_c = \sqrt{\frac{3}{2}} m_\pi. \tag{5.3}$$

Charged pions moving with the momentum $k > k_c$ will be absolutely stable. The decay (5.1) then will go in the opposite direction:

$$e^\pm \to \pi^\pm \nu. \tag{5.4}$$

Therefore, fast electrons will be converted into pions in a color superconductor.

For similar reasons, an on-shell photon can always decay into baryon number Goldstone bosons, which are exactly massless.

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Appendix A Polarization operators

There are two contributions of order μ^2 to the polarization operators defined in (4.2). One comes from momenta close to the Fermi surface: $q \equiv |\mathbf{p}| - \mu \ll \mu$. Using quasiparticle propagators (3.17) and symmetry properties (4.4), we find for the octet polarization operator in (4.2):

$$\begin{aligned} \Pi^{\mu\nu} \;=\; & \frac{i}{12} \int \frac{d^4p}{(2\pi)^4} \left\{ \frac{7J^{\mu\nu} + 2\Delta^2 I^{\mu\nu}}{[(p+k)_-^2 - \Delta^2](p_-^2 - \Delta^2)} \right. \\ & \left. + \frac{2J^{\mu\nu} + 4\Delta^2 I^{\mu\nu}}{[(p+k)_-^2 - \Delta^2](p_-^2 - 4\Delta^2)} \right\} + \delta\Pi^{\mu\nu}, \end{aligned} \tag{A.1}$$

where

$$I^{\mu\nu} = \mathrm{Sp}\, \gamma^\mu H_+ \gamma^\nu H_+, \tag{A.2}$$

$$J^{\mu\nu} = \mathrm{Sp}\, \gamma^\mu H_+ (\not{p} + \not{k})_- \gamma^\nu H_+ \not{p}_-. \tag{A.3}$$

Only positive-helicity parts of the quasiparticle propagartors, which are singular at the Fermi surface, were kept in (A.1), since typical p_0 and q that contribute to the loop integral are of order of Δ.

The term with two opposite helicity projectors, though not singular at the Fermi surface, is quadratically divergent and receives $O(\mu^2)$ contribution from the modes

with $q \sim \mu$, which means that the dependence on Δ and $k \sim \Delta$ can be omitted from this term:

$$\delta\Pi^{\mu\nu} = \frac{3i}{2}\int \frac{d^4p}{(2\pi)^4}\left(\frac{\mathrm{Sp}\,\gamma^\mu H_+ \not{p}_- \gamma^\nu H_- \not{p}_+}{p_-^2 p_+^2} - \frac{\mathrm{Sp}\,\gamma^\mu \not{p}\gamma^\nu \not{p}}{p^4}\right). \tag{A.4}$$

Subtraction of the polarization operator at zero chemical potential serves as a gauge-invariant regularization. A simple algebra yields:

$$\delta\Pi^{00} = 0, \tag{A.5}$$

$$\delta\Pi^{ij} = -\delta_{ij}\int \frac{d^3p}{(2\pi)^3}\,\frac{1}{|\mathbf{p}|}\theta(|\mathbf{p}| < \mu) = -\frac{\mu^2}{4\pi^2}\,\delta^{ij}. \tag{A.6}$$

Below, k_0 is denoted by ω and $|\mathbf{k}|$ is denoted by k. Both are much smaller than μ, and approximations like

$$|\mathbf{p}+\mathbf{k}| - \mu \approx q + \mathbf{n}\cdot\mathbf{k} \tag{A.7}$$

greatly simplify the Dirac traces and the momentum integral in (A.1). Omitting $O(\Delta/\mu)$ corrections:

$$I^{00} = 2, \qquad I^{ij} = -2n^i n^j, \tag{A.8}$$

$$J^{00} = 2\left[p_0(p_0+\omega) + q(q+\alpha)\right], \tag{A.9}$$

$$J^{0i} = 4(p_0+\omega)qn^i, \qquad J^{i0} = 4p_0(q+\alpha)n^i, \tag{A.10}$$

$$J^{ij} = 2n^i n^j\left[p_0(p_0+\omega) + q(q+\alpha)\right]. \tag{A.11}$$

Here $\alpha = \mathbf{n}\cdot\mathbf{k}$.

The momentum integral in (A.1) can be done by Wick rotation to the Euclidean space and subsequent use of the Feynman parameterization. The energy integral should be done first [9], which yields:

$$\begin{aligned}\Pi^{00} &= \frac{\mu^2}{24\pi^2 k}\int_0^1 dx\int_0^k d\alpha\,\left[9 + \frac{7x(1-x)(\alpha^2+\omega^2) - 2\Delta^2}{x(1-x)(\alpha^2-\omega^2)+\Delta^2}\right.\\ &\quad\left.+\frac{2x(1-x)(\alpha^2+\omega^2) - 4\Delta^2}{x(1-x)(\alpha^2-\omega^2)+(4-3x)\Delta^2}\right],\end{aligned} \tag{A.12}$$

$$\begin{aligned}\Pi^{0i} = \Pi^{i0} &= \frac{\mu^2\omega k^i}{12\pi^2 k^3}\int_0^1 dx\int_0^k d\alpha\,\left[\frac{7x(1-x)\alpha^2}{x(1-x)(\alpha^2-\omega^2)+\Delta^2}\right.\\ &\quad\left.+\frac{2x(1-x)\alpha^2}{x(1-x)(\alpha^2-\omega^2)+(4-3x)\Delta^2}\right],\end{aligned} \tag{A.13}$$

$$\Pi^{ij} = \frac{\mu^2}{24\pi^2 k}\int_0^1 dx\int_0^k d\alpha\left\{3\delta^{ij}+\frac{1}{2}\left[\left(1-\frac{\alpha^2}{k^2}\right)\delta^{ij}+\left(3\frac{\alpha^2}{k^2}-1\right)\frac{k^ik^j}{k^2}\right]\right.$$
$$\left.\times\left[\frac{7x(1-x)(\alpha^2+\omega^2)+2\Delta^2}{x(1-x)(\alpha^2-\omega^2)+\Delta^2}+\frac{2x(1-x)(\alpha^2+\omega^2)+4\Delta^2}{x(1-x)(\alpha^2-\omega^2)+(4-3x)\Delta^2}\right]\right\} \quad (A.14)$$

It is straightforward to check that the polarization operator is transverse.

The color-flavor algebra is simpler for singlet polarization operator, since in that case the quasiparticle in the loop belongs to a definite representation of $SU(3)$. The trace over color and flavor indices gives just the dimension of the representation:

$$K^{\mu\nu} = \frac{i}{4}\int\frac{d^4p}{(2\pi)^4}\left\{8\,\frac{J^{\mu\nu}-\Delta^2 I^{\mu\nu}}{[(p+k)_-^2-\Delta^2](p_-^2-\Delta^2)}\right.$$
$$\left.+\frac{J^{\mu\nu}-4\Delta^2 I^{\mu\nu}}{[(p+k)_-^2-4\Delta^2](p_-^2-4\Delta^2)}\right\}+3\delta\Pi^{\mu\nu}. \quad (A.15)$$

The coefficient 3 before the high-momentum contribution is due to the extra trace over flavor indices.

The momentum integration yields:

$$K^{00} = \frac{\mu^2}{8\pi^2 k}\int_0^1 dx\int_0^k d\alpha\left[9+8\,\frac{x(1-x)(\alpha^2+\omega^2)+\Delta^2}{x(1-x)(\alpha^2-\omega^2)+\Delta^2}\right.$$
$$\left.+\frac{x(1-x)(\alpha^2+\omega^2)+4\Delta^2}{x(1-x)(\alpha^2-\omega^2)+4\Delta^2}\right], \quad (A.16)$$

$$K^{0i}=K^{i0} = \frac{\mu^2\omega k^i}{4\pi^2 k^3}\int_0^1 dx\int_0^k d\alpha\left[8\,\frac{x(1-x)\alpha^2}{x(1-x)(\alpha^2-\omega^2)+\Delta^2}\right.$$
$$\left.+\frac{x(1-x)\alpha^2}{x(1-x)(\alpha^2-\omega^2)+4\Delta^2}\right], \quad (A.17)$$

$$K^{ij} = \frac{\mu^2}{8\pi^2 k}\int_0^1 dx\int_0^k d\alpha\left\{3\delta^{ij}+\frac{1}{2}\left[\left(1-\frac{\alpha^2}{k^2}\right)\delta^{ij}+\left(3\frac{\alpha^2}{k^2}-1\right)\frac{k^ik^j}{k^2}\right]\right.$$
$$\left.\times\left[8\,\frac{x(1-x)(\alpha^2+\omega^2)-\Delta^2}{x(1-x)(\alpha^2-\omega^2)+\Delta^2}+\frac{x(1-x)(\alpha^2+\omega^2)-4\Delta^2}{x(1-x)(\alpha^2-\omega^2)+4\Delta^2}\right]\right\}. \quad (A.18)$$

This polarization operator is also transverse, as it should be.

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